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C. E. Dean and M. F. Werby

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Variational methods and the derivation of shell theories to approximate vibrations of bounded elastic shells

C. E. Dean and M. F. Werby

Naval Oceanographic and Atmospheric Research Laboratory
Theoretical Acoustics Code 221
Stennis Space Center, MS 39529-5004

ABSTRACT

The calculation of vibrations, and in particular, resonances from bounded elastic shells can be quite tedious and time consuming when using the exact elastodynamic equations. Thus, a popular approach has been to employ various dynamic assumptions about the motion of the shell surface when subjected to disturbances. This can be done using variational considerations in which energy is minimized when various constraints are imposed. We exploit the technique using various assumptions which give rise to several shell theories. We can use the resulting expressions to calculate resonances over a frequency range and compare them with the exact results. We may then rank the various approximations in order of their agreement to the exact results. Limitations of each of the methods can then be outlined as well as those of shell methods in general.

KEYWORDS

Shell theory; elastic; variational methods; nontorsional

INTRODUCTION

The standard assumptions used in shell theory were formulated by A. E. H. Love (Love, 1944) and are as follows: (1) The thickness of a shell is small compared with the smallest radius of curvature of the shell; (2) The displacement is small in comparison with the shell thickness; (3) The transverse normal stress acting on planes parallel to the shell middle surface is negligible; (4) Fibers of the shell normal to the middle surface remain so after deformation and are themselves not subject to elongation. We use these assumptions in the development of a shell theory for an elastic spherical shell in the spirit of Timoshenko-Mindlin plate theory

DERIVATION OF EQUATIONS OF MOTION

In spherical shells membrane stresses (proportional to β) predominate over flexural stresses (proportional to β^2) where

$$\beta = \frac{1}{\sqrt{12}}\frac{h}{a}. \tag{1}$$

We differ from the standard derivation for the sphere (Junger and Feit, 1986) by retaining all terms of order β^2 in both the kinetic and potential energy parts of the Lagrangian. We note that this level of approximation will allow us to include the effects of rotary inertia and shear distortion in our shell theory. We begin our derivation by considering a u,v,w axis system on the middle surface of a spherical shell of radius a (measured to mid-shell) with thickness h, as shown in Fig. 1.

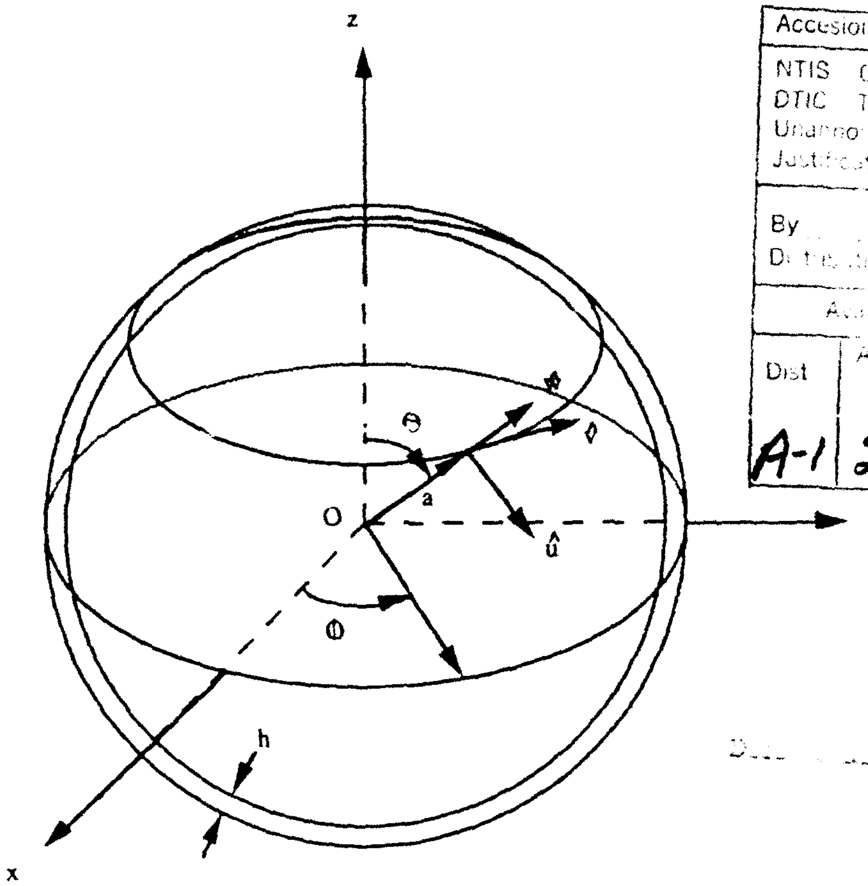

Fig. 1. Spherical shell showing the coordinate system used.

Lagrangian Variational Analysis

Thus the new Lagrangian (which is equivalent to a Timoshenko-Mindlin theory as applied to a spherical shell) is

$$L = T - V + W, \tag{2}$$

where the kinetic energy is

$$T = \frac{1}{2}\rho_s \int_0^{2\pi}\int_0^{\pi}\int_{-h/2}^{h/2} (\dot{u}_s^2 + \dot{w}_s^2)(a + x)^2, \tag{3}$$

with the surface displacements taken to be linear as in Timoshenko-Mindlin plate theory:

$$\dot{u}_s = (1 + \frac{x}{a})\dot{u} - \frac{x}{a}\frac{\partial \dot{w}}{\partial \theta}, \tag{4}$$

$$\dot{w}_s = \dot{w}. \tag{5}$$

There is no movement in the v-direction since the sound field can be assumed, without loss of generality, as torsionless. By substitution, the kinetic energy is

$$T = \pi\rho_s \int_0^{\pi} \sin\theta [(\frac{h^5}{80a^2} + \frac{h^3}{2} + ha^2)\dot{u}^2 - 2(\frac{h^5}{80a^2} + \frac{h^3}{4})\dot{u}\frac{\partial \dot{w}}{\partial \theta} + (\frac{h^5}{80a^2} + \frac{h^3}{12})(\frac{\partial \dot{w}}{\partial \theta})^2 + (\frac{h^3}{12} + ha^2)\dot{w}^2] d\theta, \tag{6}$$

or, simplifying,

$$T = \pi\rho_s h a^2 \int_0^{\pi} [(1.8\beta^4 + 6\beta^2 + 1)\dot{u}^2 - (3.6\beta^4 + 6\beta^2)\dot{u}\frac{\partial \dot{w}}{\partial \theta} + (1.8\beta^4 + \beta^2)(\frac{\partial \dot{w}}{\partial \theta})^2 + (\beta^2 + 1)\dot{w}^2]\sin\theta d\theta, \tag{7}$$

which to order β^2 is

$$T = \pi\rho_s ha^2 \int_0^\pi [(1+6\beta^2)\dot{u}^2 - 6\beta^2 \dot{u}\frac{\partial \dot{w}}{\partial\theta} + \beta^2(\frac{\partial \dot{w}}{\partial\theta})^2 + (1+\beta^2)\dot{w}^2]\sin\theta d\theta. \quad (8)$$

In a similar fashion the potential energy is

$$V = \frac{1}{2}\int_0^\pi\int_0^{2\pi}\int_{-h/2}^{h/2}(\sigma_{\theta\theta}\epsilon_{\theta\theta} + \sigma_{\phi\phi}\epsilon_{\phi\phi})(x+a)^2 \sin\theta dx d\theta d\phi, \quad (9)$$

which by substitution becomes

$$V = \frac{1}{2}\int_0^\pi\int_0^{2\pi}\int_{-h/2}^{h/2}\left[\frac{E}{1-\nu^2}\frac{1}{(x+a)^2}\left([(1+\frac{x}{a})\frac{\partial u}{\partial\theta} - \frac{x}{a}\frac{\partial^2 w}{\partial\theta^2} + w^2]\right.\right.$$
$$\left.\left.+(\cot\theta[(1+\frac{x}{a})u - \frac{x}{a}\frac{\partial w}{\partial\theta}] + w)[(1+\frac{x}{a})\frac{\partial u}{\partial\theta} - \frac{x}{a}\frac{\partial^2 w}{\partial\theta^2} + w]\right)\right](x+a)^2 \sin\theta dx d\theta d\phi. \quad (10)$$

or finally,

$$V = \frac{\pi E h}{1-\nu^2}\int_0^\pi [(w + \frac{\partial u}{\partial\theta})^2 + (w + u\cot\theta)^2 + 2\nu(w + \frac{\partial u}{\partial\theta})(w + u\cot\theta) + \beta^2[(\frac{\partial u}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2})^2 \cot^2\theta(u - \frac{\partial w}{\partial\theta})^2$$
$$+2\nu\cot\theta(u - \frac{\partial w}{\partial\theta})(\frac{\partial u}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2})]]\sin\theta d\theta, \quad (11)$$

where the nonvanishing components of the strain are

$$\epsilon_{\theta\theta} = \frac{1}{a}\left(\frac{\partial u}{\partial\theta} + w\right) + \frac{x}{a^2}\left(\frac{\partial u}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2}\right), \quad (12)$$

and

$$\epsilon_{\phi\phi} = \frac{1}{a}(\cot\theta u + w) + \frac{x}{a^2}\cot\theta\left(u - \frac{\partial w}{\partial\theta}\right), \quad (13)$$

with nonzero stress components are

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\epsilon_{\theta\theta} + \nu\epsilon_{\phi\phi}), \quad (14)$$

and

$$\sigma_{\phi\phi} = \frac{E}{1-\nu^2}(\epsilon_{\phi\phi} + \nu\epsilon_{\theta\theta}), \quad (15)$$

where E is Young's modulus.

Finally, the work done by the surrounding fluid on the sphere is

$$W = 2\pi a^2 \int_0^\pi p_a w \sin\theta d\theta, \quad (16)$$

where p_a is the pressure at the surface of the shell.

<u>Lagrangian Density and Equations of Motion</u>

Since the integration along the polar angle is intrinsic to the problem, the solution must be found using a *Lagrangian density*:

$$\mathcal{L}_1 = \pi\rho_s ha^2[(1+6\beta^2)\dot{u}^2 - 6\beta^2\dot{u}\frac{\partial\dot{w}}{\partial\theta} + \beta^2(\frac{\partial\dot{w}}{\partial\theta})^2 + (1+\beta^2)\dot{w}^2]\sin\theta - \frac{\pi E h}{1-\nu^2}[(w + \frac{\partial u}{\partial\theta})^2 + (w + u\cot\theta)^2$$
$$+2\nu(w + \frac{\partial u}{\partial\theta})(w + u\cot\theta) + \beta^2[(\frac{\partial u}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2})^2 + \cot^2\theta(u - \frac{\partial w}{\partial\theta})^2 + 2\nu\cot\theta(u - \frac{\partial w}{\partial\theta})(\frac{\partial u}{\partial\theta} - \frac{\partial^2 w}{\partial\theta^2})]]\sin\theta$$
$$+2\pi a^2 p_a w\sin\theta, \quad (17)$$

with corresponding differential equations

$$0 = \frac{\partial\mathcal{L}_1}{\partial u} - \frac{d}{d\theta}\frac{\partial\mathcal{L}_1}{\partial u_\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}_1}{\partial u_t}, \quad (18)$$

and

$$0 = \frac{\partial\mathcal{L}_1}{\partial w} - \frac{d}{d\theta}\frac{\partial\mathcal{L}_1}{\partial w_\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}_1}{\partial w_t}. \quad (19)$$

Substituting, we find

$$0 = (1+\beta^2)\left[\frac{\partial^2 u}{\partial\theta^2} + \cot\theta\frac{\partial u}{\partial\theta} - (\nu + \cot^2\theta)u\right] - \beta^2\frac{\partial^3 w}{\partial\theta^3} - \beta^2\cot\theta\frac{\partial^2 w}{\partial\theta^2}$$
$$+[(1+\nu) + \beta^2(\nu + \cot^2\theta)]\frac{\partial w}{\partial\theta} - \frac{a^2}{c_p^2}[(1+6\beta^2)\ddot{u} - 3\beta^2\frac{\partial\ddot{w}}{\partial\theta}], \tag{20}$$

and

$$-p_a\frac{(1-\nu^2)a^2}{Eh} = \beta^2\frac{\partial^3 u}{\partial\theta^3} + 2\beta^2\cot\theta\frac{\partial^2 u}{\partial\theta^2} - [(1+\nu)(1+\beta^2) + \beta^2\cot^2\theta)]\frac{\partial u}{\partial\theta}$$
$$-\beta^2\cot\theta(2-\nu+\cot^2\theta)\frac{\partial w}{\partial\theta} - 2(1+\nu)w - \frac{a^2}{c_p^2}(1+\beta^2)\ddot{w}. \tag{21}$$

Differential equations (20) and (21) have solutions of the form

$$u(\eta) = \sum_{n=0}^{\infty} U_n(1-\eta^2)^{1/2}\frac{dP_n}{d\eta}, \tag{22}$$

and

$$w(\eta) = \sum_{n=0}^{\infty} W_n P_n(\eta), \tag{23}$$

where $\eta = \cos\theta$ and $P_n(\eta)$ are the Legendre polynomials of the first kind of order n. The differential equations of motion (20) and (21) are satisfied if the expansion coefficients U_n and W_n satisfy a homogeneous system of linear equations.

Vacuum Case

If we consider the simpler vacuum case first, where $p_a = 0$, the linear equations are

$$0 = [\Omega^2(1+6\beta^2) - (1+\beta^2)\kappa]U_n - [\beta^2(\kappa - 3\Omega^2) + 1 + \nu]W_n, \tag{24}$$

and

$$0 = -\lambda_n(\beta^2\kappa + 1 + \nu)U_n + [\Omega^2(1+\beta^2) - 2(1+\nu) - \beta^2\kappa\lambda_n]W_n, \tag{25}$$

where $\Omega = \omega a / c_p$, $\kappa = \nu + \lambda_n - 1$, and $\lambda_n = n(n+1)$. The determinant of (24) and (25) yields a frequency equation of the form

$$0 = \Omega^4(1+7\beta^2) + \Omega^2\{-(1+2\beta^2)\kappa - [2(1+\nu)(1+6\beta^2) + \beta^2\kappa\lambda_n]$$
$$+3\beta^2\lambda_n(1+\nu)\} + (\lambda_n - 2)(1-\nu^2) + \beta^2\kappa[\kappa\lambda_n + 2(1-\lambda_n)(1+\nu)]. \tag{26}$$

Fluid Loaded Case

We begin consideration of the fluid loaded case by noting that "for a plate, fluid loaded on one side, of mass per unit area $\rho_s h$, the appropriate nondimensional measure of fluid loading at a frequency ω is $\rho c / \omega\rho_s h$." (Junger and Feit, 1986, p. 237). Analogously, we may expand the surface pressure for a sphere in terms of modal specific acoustic impedances z_n as follows,

$$p(a,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z_n\dot{W}_{mn}P_n^m(\cos\theta)\cos m\phi, \tag{27}$$

where

$$z_n = i\rho c\frac{h_n(ka)}{h_n'(ka)}. \tag{28}$$

Splitting z_n into real and imaginary parts, we have

$$z_n = r_n - i\omega m_n, \tag{29}$$

where

$$r_n = \rho c\,\mathrm{Re}\left\{\frac{ih_n(ka)}{h_n'(ka)}\right\}, \tag{30}$$

and

$$m_n = -\frac{\rho c}{\omega}\mathrm{Im}\left\{\frac{ih_n(ka)}{h_n'(ka)}\right\}. \tag{31}$$

For our simpler case of nontorsional ensonification, the surface pressure expansion simplifies to

$$p_n(\theta) = -\sum_{n=0}^{\infty} z_n \dot{W}_n P_n(\cos\theta), \tag{32}$$

which by substitution becomes

$$p_n(\theta) = -\sum_{n=0}^{\infty} (-i\omega W_n r_n - \omega^2 W_n m_n) P_n(\cos\theta). \tag{33}$$

Substitution of (33) into (20) and (21) will result in simultaneous linear equations of the form:

$$0 = \{\Omega^2(1+6\beta^2) - (1+\beta^2)\kappa\}U_n - \{\beta^2(\kappa - 3\Omega^2) + 1 + \nu\}W_n, \tag{34}$$

and

$$0 = -\lambda_n(\beta^2\kappa + 1 + \nu)U_n + [\Omega^2(1 + \frac{m_n}{\rho_s h} + \beta^2) + i\frac{a}{h}\frac{r_n}{\rho_s c_p}\Omega - 2(1+\nu) - \beta^2\kappa\lambda_n]W_n. \tag{35}$$

Setting the real part of the determinant of (34) and (35) to zero results in a quadratic equation in Ω^2 for the fluid loaded case. If we define

$$\alpha = \frac{m_n}{\rho_s h}, \tag{36}$$

and neglect terms of order greater than β^2, then the quadratic is, finally,

$$0 = \Omega^4(1 + \alpha + 7\beta^2 + 6\alpha\beta^2) - \Omega^2[(1 + \alpha + 2\beta^2 + \alpha\beta^2)\kappa + 2(1+\nu)(1+6\beta^2) + \beta^2\kappa\lambda_n - 3\beta^2\lambda_n(1+\nu)] \\ +3\beta^2\lambda_n(1+\nu)] + (\lambda_n - 2)(1-\nu^2) + \beta^2\kappa[\kappa\lambda_n + 2(1-\lambda_n)(1+\nu)]. \tag{37}$$

CONCLUSIONS

With (26) and (37) in hand, the obvious next step is to test the results numerically against exact results. We note that by setting β^2 to zero we revert our solutions to previously derived models (Junger and Feit, 1986). Similarly, setting α to zero in (37) is equivalent to removing the fluid loading from the model. This makes (37) revert to (26). By alternately retaining or zeroing β^2 in (26), and similarly for α in (37), we have three distinct models with differing degrees of physicality. We can use the resulting expressions to calculate resonances over a frequency range and compare them with the exact results. We may then rank the various approximations in order of their agreement to the exact results. Limitations of each of the methods can then be outlined as well as those of shell methods in general.

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REFERENCES

Junger, M.C. and D. Feit (1986). *Sound, Structures, and Their Interaction.* 2nd ed. MIT Press, Cambridge, Mass.
Love, A.E.H. (1944). *A Treatise on the Mathematical Theory of Elasticity.* Dover, New York.